Talk 4: sSet

0. Introduction

Last time we ran into two categories

$$\mathrm{dgLie}_{\mathbb{K}} \quad \& \quad \mathrm{dgCAlg}_{\mathbb{K}}$$

with forgetful functors to $\mathrm{Ch}_{\mathbb{K}} = \mathrm{dgVect}_{\mathbb{K}}$

We will say, e.g., $f: A \to A'$ in dgCAlg is a <u>weak equivalence</u> if f is a quasi-iso (i.e. a wk equivalence after forgetting to ~~dg~~Vect)

We would like to view wk equiv $f: A \xrightarrow{\simeq} A'$ — notation!

as equivalences, even though they need not admit inverses.

The "traditional" machinery of homological algebra works well for anything of the form $\mathrm{Ch}(\mathcal{A})$, with $\mathcal{A}$ an abelian category:

here I mean manipulating w/ projective/injective resolutions, spectral sequences, etc.

[I'll assume you have some practice with such manipulations even if you've never thought much about the general situation.

But dgCAlg is not at all like $Ch(\mathcal{A})$: it is far from being an additive or abelian category!

This kind of situation appears in math a lot, hence:

Vague goal: Have a theory, as convenient & clear as homological algebra with $Ch(\mathcal{A})$, for a category $\mathcal{C}$ with a subcategory $\mathcal{W}$ of "weak equivalences".

One of the first people to clearly recognize this issue & to develop a general approach was Quillen. He invented model categories, which we will discuss next week & which, while limited in applicable range, are often the most convenient when available.

Quillen was motivated by several distinct problems in the late 1960's:

① to develop a "derived functor" on CAlg of the Kähler differentials functor

$A \longmapsto \Omega^1_A$ ~ also $A \to \mathrm{Der}_{\mathrm{com}}(A, A)$ for "deformations of comm algs" (akin to Hoch*)

(2) to find an "algebraic category" encoding rational spaces: in fact, a subcategory of dgLie!
[rational homotopy theory & subject of later talks]

~~Of course~~, $\mathrm{Spaces}^{s.c.}_{\mathbb{Q}} \longrightarrow \mathrm{dgLie}_{>0}$

$\mathrm{Spaces}^{s.c.}_{\mathbb{Q}} = \left\{ X \in \mathrm{Top} : \begin{array}{l} \pi_0 X = \pi_1 X = * \\ \pi_k X \in \mathrm{Vect}_{\mathbb{Q}} \end{array} \right\}$

$\mathrm{dgLie}_{>0} = \{ L : L^k = 0,\ k \geq 0 \}$

rat htpy eq $\rightsquigarrow$ quasi-iso

(3) ~~And~~ And here's the example we'll focus on today:
Dan Kan introduced ~~an~~ category known as simplicial sets to provide a combinatorial model for homotopy theory. Quillen wanted to make this statement utterly precise.

Kan had introduced an adjunction

(1-1) — left adjoint

$$\begin{tikzcd} s\mathrm{Set} \arrow[r, bend left, ""] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, bend left, "\mathrm{sing}"] \end{tikzcd}$$

& Quillen provided a setting in which

$$s\mathrm{Set}[W^{-1}] \simeq \mathrm{Top}[W^{-1}]$$

$\rightsquigarrow$ categories

I. Simplicial sets

Caveat: Unless you're very combinatorially minded, simplicial sets may not ~~appeal~~ speak compellingly to you on first encounter. Their main selling point is that they are quite convenient.

Their current ubiquity also means that once you absorb the basics — which need not take too long — you can suddenly follow a lot of talks & texts that were inaccessible before. //

There are two approaches:

- "abstract" the idea from simplicial complexes & combinatorial topology
  concrete but a bit slow
- give a purely categorical definition & then ~~extract~~ ~~abstract~~ concrete manifestations

My preference would ordinarily be the first but given our future role for simplicial sets, the 2nd is more natural.

See Friedman for a nice ~~discussion~~ example of 1st approach

Def The *ordinal category* Δ has

- *objects*: finite, totally ordered sets
- *morphisms*: ~~order~~ order-respecting maps

Ex: $[n] := \{0 < 1 < 2 < \dots < n\}$

$$\Delta([1],[2]) = \left\{ (0\mapsto 0,\ 1\mapsto 0),\ (0\mapsto 0,\ 1\mapsto 1),\ (0\mapsto 0,\ 1\mapsto 2),\ (0\mapsto 1,\ 1\mapsto 1),\ (0\mapsto 1,\ 1\mapsto 2),\ (0\mapsto 2,\ 1\mapsto 2) \right\}$$

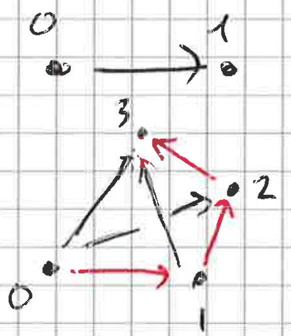
I like $\mathcal{C}(x,y)$ as notation, or $\mathrm{Hom}_{\mathcal{C}}(x,y)$ but never $\mathrm{Hom}(x,y)$!

Rmk the full subcategory of $\{[n]\}_{n\in\mathbb{N}}$ is a skeletal subcategory

You should picture $[n]$ as the categorical/combinatorial essence of an n-simplex w/ ordered vertices

$[1] \leftrightsquigarrow$

$[3] \leftrightsquigarrow$



I have hidden the "identity morphisms" (view $[n]$ as ~~poset~~ category)

Using this picture, you can see that

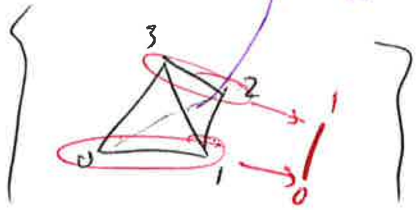
$$\Delta([1],[3]) = \left\{ \begin{pmatrix} 0 & 1 \\ i & j \end{pmatrix} : i \leq j \right\} = \text{"all 1-simplices in a 3-simplex"}$$

including the degenerate ones

What about $\Delta([3],[1])$?

"extend by linearity"

$$\left\{ \left(\begin{matrix} 0 & 1 & 2 & 3 \\ \downarrow & & \downarrow & \\ 0 & & 1 & \end{matrix} \right) \right\} \longleftrightarrow$$



determined by first # to go to 1

ways of collapsing a 3-simplex to a 1-simplex

also 

<u>Idea</u>: Δ encodes how ordered simplices relate to one another (expansions, collapses, etc)

} moduli perspective

<u>Idea</u>: In ~~algebraic topology~~, we only really care about spaces in terms of how simplices map into them, so why not <u>describe</u> spaces purely in terms of that data?

A simplicial set is the distilled essence of that philosophy.

<u>notation</u>: sSet or Set_Δ or $Set^{\Delta^{op}}$

<u>Def</u> The category of <u>simplicial sets</u> is $\mathrm{Fun}(\Delta^{op}, \mathrm{Set})$.

A <u>simplicial set</u> is thus a functor

$$X : \Delta^{op} \longrightarrow \mathrm{Sets}.$$

"a simplicial moduli problem"

Can replace Sets by $\mathcal{C}$ to get "simplicial objects in category $\mathcal{C}$"

II Geometric realization

There's a lot of structural stuff to do with sSet but I want to provide some geometry as soon as possible.

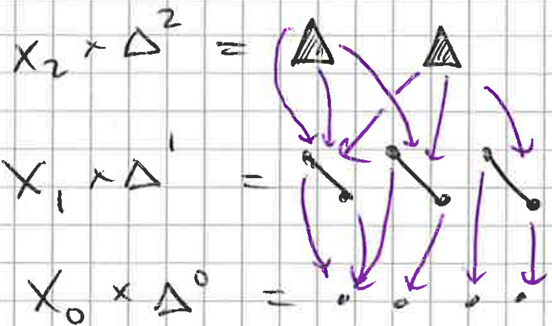
<u>Idea</u>: view a simplicial set X as the "instructions" for assembling a space $|X|$

= convex hull of $n+1$ ~~linearly~~ affinely independent points

Let $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1\}$ denote the <u>standard n-simplex</u> with its induced topology

<u>Picture</u> $\Delta^0 = \bullet \quad \Delta^1 = \bullet\!\!-\!\!\bullet \quad \Delta^2 =$  ...

Then interpret $X_n := X([n])$ as labeling a set of n-simplices



$\curvearrowleft$ = this face identifies w/ this lower-dim simplex

We need to know how to $\boxed{\text{glue}}$ them together

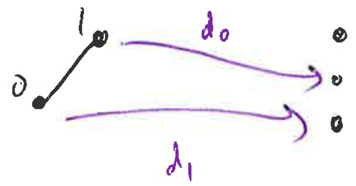
This data is included in X via "structure maps"

Eg. $\Delta([0],[1]) = \Big\{ \underset{d^1}{0 \mapsto 0 \quad 0\!-\!1}, \quad \underset{d^0}{0 \mapsto 1 \quad 0\!-\!1} \Big\}$ "coface maps"

$\Downarrow$

$$\begin{tikzcd} X([1]) \arrow[r, shift left, "d_1 = (d^1)^*"] \arrow[r, shift right, "d_0"'] & X([0]) \\ X_1 \arrow[u, equal] & X_0 \arrow[u, equal] \end{tikzcd}$$

$d_0 = (d^0)^*$



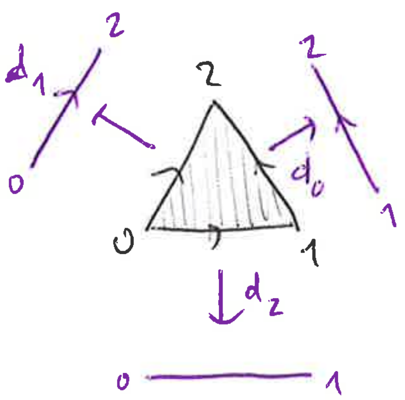
$$\Delta([n],[n+1]) = \left\{ d^i : j \longmapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \right\}_{i=0}^{n}$$

d^i skips the i^{th} vertex

$\Downarrow$

$$\begin{tikzcd} X_{n+1} \arrow[r, bend left, "d_0 = (d^0)^*"] \arrow[r, "d_1"] \arrow[r, phantom, shift right=2, "\vdots"] \arrow[r, bend right, "d_n"'] & X_n \end{tikzcd}$$

$d_i(x)$ says what ~~the~~ is the i^{th} face of an $(n+1)$-simplex $x \in X_{n+1}$



We now use this 'information' to define the gluing:

$\boxed{\text{V.1.}}$ if $x \in X_{n+1}$ & $y = d_i(x)$, then attach $y \times \Delta^n$ as the i^{th} face of the $(n+1)$-simplex $x \times \Delta^{n+1}$

gluing = "impose equivalence relation"
= colimit

$\boxed{\text{V.2}}$ in formulas

$$|X| := \left(\coprod_n X_n \times \Delta^n \right) \Big/ \underbrace{\left(d_i(x), (a_0, \dots, a_n)\right)}_{\in\, X_n \times \Delta^n} \sim \underbrace{\left(x, (a_0 \cdots a_{i-1}\, \overset{i^{th}\text{ spot}}{0}\, a_i \cdots a_n)\right)}_{\in\, X_{n+1} \times \Delta^{n+1}}$$

Example / exercise

Let
$$\Delta[n] : \Delta^{op} \longrightarrow \mathrm{Set}$$
$$m \longmapsto \Delta(m, n)$$
be the Yoneda embedding.

Then $|\Delta[n]| \cong \Delta^n$.

You should also take your favorite simplicial complex & encode it as a simplicial set — if you have one!

Def The functor

$$sSet \xrightarrow{\ |-|\ } Top$$
$$X \longmapsto |X|$$

is the <u>geometric realization</u>

Lemma

The geometric realization is left adjoint to the singular set functor

$$Sing : Top \longrightarrow sSet$$
$$Y \longmapsto \big([n] \mapsto Top(\Delta^n, Y)\big)$$

$\|$ continuous maps $\Delta^n \longrightarrow Y$

Pf ~~[illegible]~~

Any functor $X : \Delta^{op} \to Set$ is the colimit of representable functor

$$X \cong \operatorname*{colim}_{\Delta[n] \to X} \Delta[n]$$

colimits commute

& so

$$Top(|X|, Y) \cong Top\big(\operatorname*{colim}_{\Delta[n] \to X} |\Delta[n]|, Y\big)$$
$$\cong \lim_{\overleftarrow{\Delta[n] \to X}} sSet(\Delta[n], Sing\, Y)$$

$\cong \mathrm{sSet}(\cancel{\Delta[n]}, \mathrm{Sing}\, Y)$. $\square$

Rmk $|X|$ is always a CW-cplx & hence a compactly generated Hausdorff space. It can be convenient to work w/ CGHaus rather than Top //

Other facts

$|\mathrm{Sing}\, X| \longrightarrow X$ is a wk htpy equivalence

Anticipatory remark:

This adjunction lets us transport ideas & constructions btw sSet & Top.

In particular, we say

$$X \xrightarrow[\simeq]{f} Y \quad \text{in sSet}$$

if

$$|X| \xrightarrow[\simeq]{|f|} |Y| \quad \text{in Top}$$

mention "simplicial n-sphere" & horns

weak homotopy equivalence, i.e.

$$\forall p \in |X|, \quad \pi_n|f| : \pi_n(|X|, p) \xrightarrow{\cong} \pi_n(|Y|, |f|(p))$$

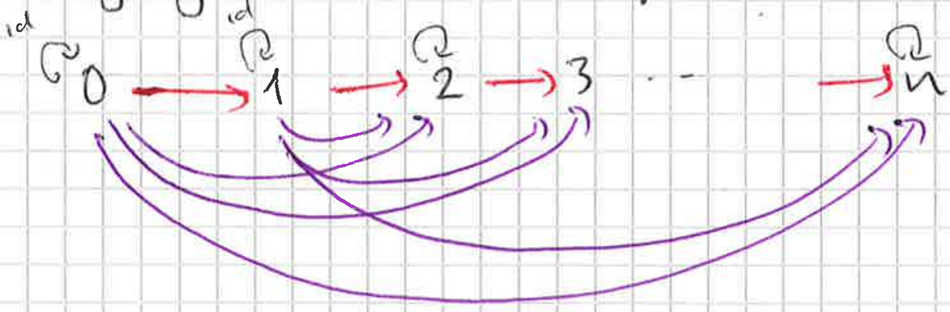
Kan & others developed analogs in sSet of all the usual notions in topology, like (co)fibration, htpy groups, etc.

(W)

III. Relationship with categories

Simplicial sets are important because they provide a useful perspective on other kinds of situations. In a moment, we'll touch on chain complexes but let's start with categories.

Observation: Each $[n]$ can be viewed as a category with $n+1$ objects



$\circlearrowleft$ = identity
$\longrightarrow$ = generating
$\longrightarrow$ = composite

Hence

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d] & \mathrm{Cat} \arrow[dl, dashed, "N"] \\ \mathrm{sSet} & \end{tikzcd}$$

N = ~~left Kan extension~~ "nerve"

$$N\mathcal{C}_n = \mathrm{Cat}([n], \mathcal{C})$$

$$= \left\{ \begin{array}{c} \text{all composable } n\text{-chains} \\ c_0 \to c_1 \to \cdots \to c_n \quad \text{in } \mathcal{C} \end{array} \right\}$$

~~The "face maps" $d_i : N\mathcal{C}_{n+1} \to N\mathcal{C}_n$ send $(c_0 \to c_1 \to \cdots \to c_n)$ to $(c_0 \to \cdots \to c_{i-1} \xrightarrow{\text{composite}} c_{i+1} \to \cdots \to c_n)$~~

The "face maps" $d_i : N\mathcal{C}_{n+1} \to N\mathcal{C}_n$ is

– if $i = 0, n$, drop the end e.g.

$$(c_0 \to c_1 \to \cdots \to c_n) \overset{d_0}{\longmapsto} (c_1 \to \cdots \to c_n)$$

– if $0 < i < n$, compose the maps

$$(c_0 \to \cdots \to c_{i-1} \xrightarrow{a} c_i \xrightarrow{b} c_{i+1} \to \cdots \to c_n)$$

$$\overset{d_i}{\longmapsto} (c_0 \to \cdots \to c_{i-1} \xrightarrow{b \circ a} c_{i+1} \to \cdots \to c_n)$$

<u>Ex</u>: There is a left adjoint

$$\tau_1 : sSet \longrightarrow Cat$$

τ_1 for "first truncation"

Describe it!

You can picture the nerve $N\mathcal{C}$ as a kind of "graph visualization" of $\mathcal{C}$

Each n-simplex $N\mathcal{C}_n$ encodes an ordered path through $\mathcal{C}$ & indicates all compositions

<u>Ex</u> If $F, G : \mathcal{C} \to \mathcal{D}$ are functors & $\eta : F \Rightarrow G$ is a natural transformation, then $|NF|, |NG| : |N\mathcal{C}| \to |N\mathcal{D}|$ are homotopic

at end

IV Relationship with chain complexes

Let's finally explain how sSet provide a "nonabelian" version of homological algebra by seeing ~~that~~ ordinary homological algebra relates.

<u>Def</u> A <u>simplicial abelian group</u> is a functor $A : \Delta^{op} \to \text{Ab Gps}$.

Its <u>Moore complex</u> $C_* A_* \in Ch_{\geq 0}(\text{Ab})$ has

$$C_n A_* = A_n$$

and differential

$$\partial : C_n A \to C_{n-1} A$$

$$a \longmapsto \sum_{i=0}^{n} (-1)^i d_i(a)$$

(the fact that $\partial^2 = 0$ follows from "simplicial identities")

~~below $d_i d_j = d_j d_i$~~

<u>Ex</u> The composite functor

$$\text{Top} \xrightarrow{\text{Sing}} \text{sSet} \xrightarrow[\mathbb{Z}]{\text{Free}_{\mathbb{Z}}} \text{sAb} \xrightarrow{\text{Moore}} Ch_{\geq 0}$$

$$X \longmapsto \text{Sing}\, X \longmapsto \mathbb{Z}\,\text{Sing}\, X \longmapsto (\mathbb{Z}\,\text{Sing}\, X)_*$$

is the <u>singular chain complex</u> of X.

You can view simplicial constructions as factoring this construction into 'atomic' components.

A different associated chain complex will be more convenient.

<u>Def</u> The <u>normalized chain complex</u> N_*A_* of a simpl. abelian group A has

$$N_nA = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n$$

and differential

$$\partial = (-1)^n d_n : N_nA \longrightarrow N_{n-1}A .$$

<u>Rmk</u> The inclusion $N_*A \hookrightarrow C_*A$ is a chain homotopy equivalence & hence a quasi-isomorphism. //

The crucial result is the following

<u>Thm</u> (Dold-Kan correspondence)

(1) The functor $N_* : sAb \longrightarrow Ch_{\geq 0}(Ab)$ is an equivalence of categories.

(2) Moreover, $\pi_n A \cong H_n(N_*A) \cong H_n(C_*A)$

e.g. $\pi_n(|A|)$

Remark

The proof of (1) is not bad, although a little involved combinatorially.

The inverse is a simpl. ab gp $DK(C_*)$ with

$$DK_n(C_*) = \bigoplus_{\alpha : [n] \twoheadrightarrow [k]} C_k$$

and the following maps:

for $\beta : [n'] \longrightarrow [n]$ in Δ,

$$\beta^* : DK_n(C_*) \longrightarrow DK_{n'}(C_*)$$

is given by a matrix $\left(f_{\alpha,\alpha'} : C_k \to C_{k'}\right)_{\substack{\alpha : [n] \twoheadrightarrow [k] \\ \alpha' : [n'] \twoheadrightarrow [k']}}$

where • if

$$\begin{tikzcd} {[n']} \arrow[r, "\beta"] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\circlearrowleft"] & {[n]} \arrow[d, "\alpha"] \\ {[k']} \arrow[r, equal, "id"] & {[k']} \end{tikzcd}$$

then $f_{\alpha\alpha'} = id$

• if $k' = k-1$ and

$$\begin{tikzcd} {[n']} \arrow[r, "\beta"] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\circlearrowleft"] & {[n]} \arrow[d, "\alpha"] \\ {[k']} \arrow[r] & (1<2<\dots<n) \subset [k] \end{tikzcd}$$

then $f_{\alpha\alpha'} = d_k$

• $f_{\alpha\alpha'} = 0$ else